On the Complexity of Counting Irreducible Components and Computing Betti Numbers of Algebraic Varieties

Abstract

In the first part of this thesis we give a uniform method for the two problems $\#\mathrm{CC}_{\mathbb{C}}$ and $\#\mathrm{IC}_{\mathbb{C}}$ of counting the connected and irreducible components of complex algebraic varieties, respectively. Our algorithms are purely algebraic and work in parallel polynomial time, i.e., they can be implemented by algebraic circuits of polynomial depth.

The second part contains lower bounds in terms of hardness results. In particular, we show that the problem of deciding connectedness of a complex affine or projective variety given over the rationals is PSPACE-hard. We further extend this result to higher Betti numbers.

In the third part we study the dependency of the complexity of $\#\mathrm{IC}_{\mathbb{C}}$ on its combinatorial parameters. The crucial complexity parameter for the problem turns out to be the number of equations. This fact is illustrated by our result that one can count the absolutely irreducible factors of a multivariate polynomial in parallel polylogarithmic time.

Furthermore, we show that one can solve $\#\mathrm{IC}_{\mathbb{C}}$ for a fixed number of equations in the BSS-model in polynomial time, and in the Turing model in randomised parallel polylogarithmic time. These results hold also for polynomials given by straight-line programs using their length and the degree as input parameters.